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THE DISTRIBUTION OF FIRST ENTRY TIME WITH APPLICATIONS TO RUIN PROBABILITIES

by

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Abstract

The present paper tries to establish differential equations for identifying the distribution of the first entry time to a Borel set for some Markov process. Applications are given to cases where the process is of pure diffusion type and consisting of a diffusion and jump part. The starting point is to define a process measuring conditional probabilities of entry as time varies over some fixed interval. Stopping this process at the time of entry, it becomes a martingale. Putting structure on the process such that Ito's formula becomes applicable, we can establish (partial) differential equations for evaluating the probability of first entry to some Borel set for the process of interest. In particular, this opens the possibility of studying ruin probabilities for fairly general models and boundary crossing probabilities for Brownian motion when the boundary is an absolutely continuous function.

Keywords: Ito diffusion, Ito lemma, Stochastic interest, Boundary crossing probabilities, Point process.

1 Introduction

This paper is motivated by the martingale approach in Møller (1993), where differential equations for evaluating the probability of ruin were established by considering a process arising as conditional probabilities. As outlined there this approach seems to be new. An alternative approach can be seen in Dassios and Embrechts (1989).

In Møller (1993) attention was only given to PD (piecewise deterministic) Markov processes. We will here formulate the approach more generally starting by introducing the first time of entry to some Borel set for a general Markov process. We can hereby suggest a fairly general approach for studying the probability of ruin and boundary crossing probabilities for Brownian motion as solutions to differential equations.

In Section 2, we outline the basic theory and formulate the general martingale property. In Subsection 3.1 we consider a real-valued Ito diffusion process, and by use of Ito's formula we obtain differential equations for evaluating the probability of first entry to some Borel set.

In Subsection 3.2 we extend the model in Subsection 3.1 by adding a jump process to obtaining a 'piecewise diffusion' process (risk business). We give an example of a differential equation for evaluating the probability of ruin in an economic environment, where the claim amounts are i.i.d. (independent and identically distributed) and exponentially distributed and the force of interest (force of return on an investment) is of Gaussian type.

2 The basic martingale property

Assume there is given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where the space $(\Omega, \mathcal{F}, P)$ is assumed to be complete, and the family $(\mathcal{F}_t)_{t\geq 0}$ of σ-algbras satisfies the usual conditions.

Let $Y = (Y_t)_{t \geq 0}$ be an adapted ($\mathcal{F}_t$-measurable) cadlag (right-continuous paths with left-hand limits) process, taking values in $\mathcal{R}^n$, the n-dimensional euclidian space. The Borel σ-algebra on $\mathcal{R}^n$ is denoted $\mathcal{B}^n$. For a set $F \in \mathcal{B}^n$, we let F^c denote its complement.

We could allow Y_t to take values in a general topological space, but to make the presentation concise we avoid this. The first entry time

$$\tau_D = \inf\{t \geq 0 \,|\, Y_t \in D\}, \quad D \in \mathcal{B}^n,$$

is then a Markov time, that is, $\{\tau_D \leq t\} \in \mathcal{F}_t$. We assume now that Y_t is an $\mathcal{F}_t$-Markov process, that is, $\sigma(Y_s,\, s \geq t)$ and $\mathcal{F}_t$ are independent given Y_t. We fix a period of time $T \leq \infty$ and define the conditional probabilities

$$\Psi(t, y) = P(\inf_{t \leq s < T}\{Y_s \in D\} \,|\, Y_t = y), \quad y \in \mathcal{R}^n.$$

By the Markov property

$$\Psi(t, Y_t) = P(\inf_{t \leq s < T}\{Y_s \in D\} \,|\, \mathcal{F}_t),$$

and furthermore, by definition,

$$\Psi(t, y) = 1, \quad y \in D, \quad t \in [0, T). \tag{2.1}$$

For any $t \in [0, T)$ we write

$$\begin{aligned} I(\tau_D < T) &= I(\tau_D \leq t) + I(t < \tau_D < T) \\ &= I(\tau_D \leq t) + I(\tau_D > t) I(\inf_{t \leq s < T}\{Y_s \in D\}). \end{aligned} \tag{2.2}$$

Defining $M_t = P(\tau_D < T \,|\, \mathcal{F}_t) = E(I(\tau_D < T) \,|\, \mathcal{F}_t)$, and taking conditional expectation w.r.t. $\mathcal{F}_t$ in (2.2) and using the Markov property, we get

$$\begin{aligned} M_t &= I(\tau_D \leq t) + P(t < \tau_D < T \,|\, \mathcal{F}_t) \\ &= I(\tau_D \leq t) + I(\tau_D > t) P(\inf_{t \leq s < T}\{Y_s \in D\} \,|\, \mathcal{F}_t) \\ &= I(\tau_D \leq t) + I(\tau_D > t)\Psi(t, Y_t). \end{aligned} \tag{2.3}$$

If Ψ is a continuous function from $\mathcal{R}_+ \times \mathcal{R}^n$ to $\mathcal{R}_+$, then M_t becomes cadlag with left-hand limits

$$M_{t-} = I(\tau_D < t) + I(\tau_D \geq t)\Psi(t, Y_{t-}).$$

In the sequel we will always assume that Ψ is chosen such that M_t becomes cadlag.

Inserting $t \wedge \tau_D$ in (2.3), we obtain that

$$\begin{aligned} M_{t \wedge \tau_D} &= I(\tau_D \leq t) + I(\tau_D > t)\Psi(t \wedge \tau_D, Y_{t \wedge \tau_D}) \\ &= \Psi(t \wedge \tau_D, Y_{t \wedge \tau_D}), \quad t \in [0, T], \end{aligned} \tag{2.4}$$

where the first equality sign follows by $I(\tau_D \leq t \wedge \tau_D) = I(\tau_D \leq t)$, and the second by (2.1): $\Psi(\tau_D, Y_{\tau_D}) = 1$, $\tau_D < T$, and in the case $\tau_D \geq T$, (2.4) is trivially satisfied. Thus we have obtained (optional sampling) that $\Psi(t \wedge \tau_D, Y_{t \wedge \tau_D})$ becomes a (bounded) $\mathcal{F}_t$-martingale over $[0, T]$.

Another relation: Using the optional sampling theorem, we obtain by taking conditional expectation on both sides in (2.4) w.r.t. the $\mathcal{F}_{t \wedge \tau_D}$-measurable stochastic variable $(t \wedge \tau_D, Y_{t \wedge \tau_D})$, that

$$P(\tau < T \,|\, t \wedge \tau_D, Y_{t \wedge \tau_D}) = \Psi(t \wedge \tau_D, Y_{t \wedge \tau_D}).$$

More generally: Fix an arbitrary $t' \in [0,T)$ and define

$$\tau'_D = \inf\{t \geq t' | Y_t \in D\},$$

which is the first time of entry after t', and repeat the steps above to obtain that

$$M'_{t\wedge\tau'_D} = \Psi(t \wedge \tau'_D, Y_{t\wedge\tau'_D}), \quad t \in [t', T], \tag{2.5}$$

becomes an $\mathcal{F}_t$-martingale, where

$$\begin{aligned} M'_t &= P(t' < \tau'_D < T \,|\, \mathcal{F}_t) \\ &= I(\tau'_D \leq t) + I(\tau'_D > t)\Psi(t, Y_t), \quad t \in [t', T]. \end{aligned}$$

The martingale (2.5) is useful for deriving differential equations for evaluating $\Psi(t,y)$. However, to obtain useful equations, more structure on Y_t is required:

3 Applications

3.1 Diffusions (boundary crossing probabilities)

Firstly, we assume that Y_t is a real-valued diffusion process, which in its general form appears as solution to the stochastic differential equation

$$Y_t = Y_0 + \int_0^t b(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dB_s, \tag{3.1}$$

where the coefficients $b : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ and $\sigma : \mathcal{R}_+ \times \mathcal{R} \to \mathcal{R}$ are Borel-measurable mappings, and $(B_t)_{t\geq 0}$ is a Brownian motion assumed to be given in advance on $(\Omega, \mathcal{F}, P)$. Under suitable conditions on b and σ, see e.g. Øksendal (1992, pp. 48-49), and for a given random variable U, $E[U^2] < \infty$, independent of $\mathcal{F}^B_\infty = \sigma(B_s,\ s < \infty)$, there exists a (strong) solution to (3.1) with $Y_0 = U$, which is adapted to the filtration $\mathcal{F}^B_t = \sigma(B_s,\ s \leq t)$. For the cases to be studied, we will always assume that there exsist a unique solution to (3.1), such that

$$\eta_t = \int_0^t \sigma(s, Y_s)dB_s, \tag{3.2}$$

is a well defined zero mean $\mathcal{F}^B_t$-martingale satisfying the Ito isometry (Øksendal 1992, pp. 18-21)

$$E\left[\int_0^t \sigma(s, Y_s)dB_s\right]^2 = E\left[\int_0^t \sigma^2(s, Y_s)ds\right], \quad t \in [0,T].$$

The Ito process has continuous paths of unbounded variation over any finite interval, and its bracket process is given as

$$< Y >_t = \int_0^t \sigma^2(s, Y_s)ds.$$

If σ is independent of Y_t, we obtain in particular that η_t becomes a Gaussian process, implying that η_t for each t is normally distributed with mean zero and variance

$$Var[\eta_t] = \int_0^t \sigma^2(s)ds.$$

We now introduce the technique that leads to the differential equations. We assume that $\Psi(t,y)$ has continuous partial derivatives for $t \in (0,T)$ and $y \in D^c$, which are denoted $\frac{\partial\Psi}{\partial t}(t,y)$ and $\frac{\partial\Psi}{\partial y}(t,y)$, respectively, and furthermore we assume that $\frac{\partial^2\Psi}{\partial y^2}$ exists and is continuous. We also assume that the mappings $\sigma(t,y), b(t,y)$ are piecewise continuous. Ito's lemma yields for $t \in [t', T)$ that

$$
\begin{aligned}
&\Psi(t\wedge\tau'_D, Y_{t\wedge\tau'_D}) - \Psi(t', Y_{t'}) \\
&= \int_{t'}^{t\wedge\tau'_D} \frac{\partial\Psi}{\partial t}(s,Y_s)ds + \int_{t'}^{t\wedge\tau'_D} \frac{\partial\Psi}{\partial y}(s,Y_s)\left[b(s,Y_s)ds + \sigma(s,Y_s)dB_s\right] \\
&+\frac{1}{2}\int_{t'}^{t\wedge\tau'_D} \sigma^2(s,Y_s)\frac{\partial^2\Psi}{\partial y^2}(s,Y_s)ds. \qquad (3.3)
\end{aligned}
$$

By (3.2), the continuity of $\frac{\partial\Psi}{\partial y}(t,y)$ and the left-continuity of $I(\tau_D \geq t)$, we obtain that

$$
M_t^* = \int_{t'}^{t} I(\tau'_D \geq s)\sigma(s,Y_s)\frac{\partial\Psi}{\partial y}(s,Y_s)dB_s, \quad t\in[t',T), \qquad (3.4)
$$

becomes a zero mean $\mathcal{F}_t^B$-martingale. Using that $\Psi(t\wedge\tau'_D, Y_{t\wedge\tau'_D})$ is an $\mathcal{F}_t^B$-martingale, we then get that

$$
\begin{aligned}
&\Psi(t\wedge\tau'_D, Y_{t\wedge\tau'_D}) - \Psi(t', Y_{t'}) - M_t^* \\
&= \int_{t'}^{t\wedge\tau'_D} \frac{\partial\Psi}{\partial t}(s,Y_s)ds + \int_{t'}^{t\wedge\tau'_D} b(s,Y_s)\frac{\partial\Psi}{\partial y}(s,Y_s)ds + \frac{1}{2}\int_{t'}^{t\wedge\tau'_D} \sigma^2(s,Y_s)\frac{\partial^2\Psi}{\partial y^2}(s,Y_s)ds \qquad (3.5)
\end{aligned}
$$

is a zero mean $\mathcal{F}_t^B$-martingale. Obviously (3.5) is continuous and of bounded variation, which implies that it is constant and therefore equal to zero, see Chung and Williams (1990, pp. 87-88) or Rogers and Williams (1987, p. 54). Thus for $t \in [t', T]$

$$
\int_{t'}^{t\wedge\tau'_D} \frac{\partial\Psi}{\partial t}(s,Y_s)ds + \int_{t'}^{t\wedge\tau'_D} b(s,Y_s)\frac{\partial\Psi}{\partial y}(s,Y_s)ds = -\frac{1}{2}\int_{t'}^{t\wedge\tau'_D} \sigma^2(s,Y_s)\frac{\partial^2\Psi}{\partial y^2}(s,Y_s)ds. \qquad (3.6)
$$

These arguments lead to:

Theorem 3.1 *For any fixed $t' \in [0,T)$, the process $\Psi(t\wedge\tau'_D, R_{t\wedge\tau'_D})$, $t\in[t',T)$, is a (bounded) martingale, and over the continuity points of $\sigma(t,y)$ and $b(t,y)$, the function $\Psi(t,y)$ satisfies the partial differential equation*

$$
\frac{\partial\Psi}{\partial t}(t,y) + b(t,y)\frac{\partial\Psi}{\partial y}(t,y) = -\frac{1}{2}\sigma^2(t,y)\frac{\partial^2\Psi}{\partial y^2}(t,y), \quad t\in(0,T),\ y\in D^c. \qquad (3.7)
$$

Proof: Follows by (3.6) since t' is arbitrarily chosen on $[0,T)$ and $\tau'_D \geq t'$ by definition. □

In particular, we obtain that $\Psi(t, Y_t)$, for $Y_t \in D^c$, satisfies $\mathcal{A}\Psi = 0$, where $\mathcal{A}$ is the extended generator of Y_t, but note that $\Psi(t,Y_t)$ is not a martingale.

In the following we pay attention to the homogeneous case. This is obtained by assuming that $b(t,y)$, $\sigma(t,y)$, are independent of t. Using the property of homogeneity, we get

$$
\begin{aligned}
\Psi(t,y) &= P(\inf_{t\leq s<T}\{Y_s \in A\} \mid Y_t = y), \\
&= P(\inf_{0\leq s<T-t}\{Y_s \in A\} \mid Y_0 = y) \\
&= P(\tau_D < T-t \mid Y_0 = y).
\end{aligned}
$$

We can then as well consider

$$\Psi^*(t,y) = P(\tau_D < t \,|\, Y_0 = y),$$

and since $\Psi^*(t,y) = \Psi(T-t,y)$ on $(0,T]$, we can then by virtue of Theorem 3.1 state:

Corollary 3.2 *Suppose Y_t is a homogeneous Ito process. Then over the continuity points of $\sigma(y)$ and $b(y)$, the function $\Psi^*(t,y)$ satisfies the partial differential equation*

$$-\frac{\partial \Psi^*}{\partial t}(t,y) + b(y)\frac{\partial \Psi^*}{\partial y}(t,y) = -\frac{1}{2}\sigma^2(y)\frac{\partial^2 \Psi^*}{\partial y^2}(t,y), \quad t > 0,\ y \in D^c. \tag{3.8}$$

If $T = \infty$ and Y_t is a homogeneous Markov process, we obtain by definition that $\Psi(t,y)$ becomes independent of t, which we denote by $\Psi(y)$, and is also obtained by $\Psi(y) = \lim_{t\uparrow\infty} \Psi^*(t,y)$. We can then state:

Corollary 3.3 *Suppose Y_t is a homogeneous Ito process. Then $\Psi(Y_{t\wedge\tau_D})$ is a (bounded) martingale, and over the continuitiy points of $\sigma(y)$ and $b(y)$, the function $\Psi(y)$ satisfies the differential equation*

$$b(y)\frac{d\Psi}{dy}(y) = -\frac{1}{2}\sigma^2(y)\frac{d^2\Psi}{dy^2}(y), \quad y \in D^c. \tag{3.9}$$

Since $\Psi(t\wedge\tau'_D, Y_{t\wedge\tau'_D})$ is a martingale w.r.t. the filtration generated by the Brownian motion, it has a representation of the form (see e.g. Revuz and Yor 1991, p. 187):

$$\Psi(t\wedge\tau'_D, Y_{t\wedge\tau'_D}) = \Psi(t', Y_{t'}) + \int_{t'}^{t} H(s)dB_s, \quad t \in [t',T],$$

where H is some $\mathcal{F}_t^B$-predictable process. Using (3.4) and the fact that the martingale in (3.5) is zero, we get

$$\Psi(t\wedge\tau'_D, Y_{t\wedge\tau'_D}) = \Psi(t', Y_{t'}) + \int_{t'}^{t} I(\tau'_D \geq s)\sigma(s,Y_s)\frac{\partial\Psi}{\partial y}(s,Y_s)dB_s, \quad t \in [t',T].$$

Using the boundary condition

$$\Psi(t,y) = 1, \quad t \in [0,T), \quad y \in \partial D^c, \tag{3.10}$$

where ∂D^c is the boundary of D^c, and using for $T < \infty$ the initial condition

$$\Psi(T,y) = 0, \quad y \in D^c, \tag{3.11}$$

we can hope to identify Ψ from the differential equations above.

Example 3.1. Passage time for Gaussian process. Consider the Gaussian process

$$Y_t = Y_0 + \int_0^t \sigma(s)dB_s,$$

where $\sigma(t)$ is assumed to be continuous. Let τ_{ab} be the time of first entry for Y_t to the set

$$D = (-\infty, a] \cup [b, \infty), \quad -\infty < a < b < \infty.$$

Thus

$$Q(t,y) = P(\tau_{ab} \leq t \,|\, Y_0 = y), \quad y \in (a,b),$$

is the probability that the Gaussian process will exit the interval (a,b) before time t when it starts at $y \in (a,b)$ at time zero.

Using (3.7) we can identify $Q(T,y)$ by solving the partial differential equation

$$\frac{\partial\Psi}{\partial t}(t,y) = -\frac{1}{2}\sigma^2(t)\frac{\partial^2\Psi}{\partial y^2}(t,y), \quad t \in (0,T),\ y \in (a,b),$$

under the boundary conditions

$$\Psi(t,a) = \Psi(t,b) = 1, \quad t \in (0,T),$$

and the initial condition

$$\Psi(T,y) = 0, \quad y \in (a,b).$$

There seems to be no tractable expression for the solution, but using the method of Fourier series, it can be represented as

$$\Psi(t,y) = 1 - \sum_{m=1}^{\infty} C_m \exp\left(-\frac{m^2\pi^2 \int_t^T \sigma^2(s)ds}{2(b-a)^2}\right) \sin\left[\frac{m\pi(b-y)}{b-a}\right],$$

where C_m are determined by the initial condition and hence are coefficients in the sinus series expanding 1. Thus

$$\begin{aligned} C_m &= \frac{2}{\pi}\int_0^{\pi} \sin(mx)dx \\ &= \frac{2}{m\pi}\{1-\cos(m\pi)\}. \end{aligned}$$

Hence for any $T < \infty$,

$$Q(T,y) = 1 - \sum_{m=1}^{\infty} C_m \exp\left(-\frac{m^2\pi^2 \int_0^T \sigma^2(s)ds}{2(b-a)^2}\right) \sin\left[\frac{m\pi(b-y)}{b-a}\right]. \tag{3.12}$$

The case $Y_t = B_t$ ($\sigma(t) \equiv 1$) is classical and appears in e.g. Ito and McKean (1965, pp. 30-31), who used a different approach. Anyway, (3.12) seems at least obtainable by combining this classical result with an appropriate change of time. □

Example 3.2. Boundary crossing of Brownian motion. Consider the diffusion

$$Y_t = Y_0 + \beta t - B_t,$$

and define

$$\tau = \inf\{t \geq 0 \,|\, Y_t \leq 0\}, \tag{3.13}$$

which is the first entry time of Y_t to $D = \{y \,|\, y \leq 0\}$.

Let β be a constant and let $\mathcal{N}(x)$ denote the standard normal distribution function. Consider for any $t \geq 0$, the boundary crossing probability

$$Q(t,y) = P(\inf_{0\leq s\leq t}\{B_s \geq y + \beta s\} \,|\, B_0 = 0),$$

which is equivalent to

$$Q(t,y) = P(\tau \leq t \,|\, Y_0 = y).$$

It is a well-known result that (see e.g. Lerche, 1986, p. 27)

$$Q(t,y) = 1 - \mathcal{N}\left(\frac{\beta t + y}{\sqrt{t}}\right) + e^{-2\beta y}\mathcal{N}\left(\frac{\beta t - y}{\sqrt{t}}\right), \tag{3.14}$$

with the convention $Q(t,y) = 1$ for $y < 0$ and $t \geq 0$. Also we see that $Q(t,0) = 1$, $t > 0$. It is readily checked that Q satisfies (3.8) for $t, y > 0$, with $b(y) = \beta$ and $\sigma(y) = -1$. Furthermore, defining $Q(y) = \lim_{t\to\infty} Q(t,y)$, we obtain for $\beta > 0$ that

$$Q(y) = e^{-2\beta y},$$

which obviously satisfies (3.9) with the same choice of $b(y)$ and $\sigma(y)$ as above, and becomes equivalent to

$$Q(y) = P(\tau < \infty \,|\, Y_0 = y)$$

under the convention $Q(y) = 1$ for $\beta, y < 0$. Furthermore, it is well-known that

$$Q(Y_{t\wedge\tau}) = e^{2\beta(B_{t\wedge\tau} - \beta\, t\wedge\tau - Y_0)},$$

is a martingale. □

Example 3.3. Consider the homogeneous diffusion

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t,$$

and assume that b and σ are bounded functions and that σ is non-vanishing. It then follows by Revuz and Yor (1991, pp. 278-289) with τ defined as in (3.13) that

$$P(\tau < \infty \,|\, Y_0 = y) = \frac{\int_y^\infty e^{-2\int_0^s b(u)\sigma^{-2}(u)du} ds}{\int_0^\infty e^{-2\int_0^s b(u)\sigma^{-2}(u)du} ds}, \quad y \geq 0, \tag{3.15}$$

which also is obtained by solving (3.9) under the boundary conditions $\Psi(0) = 1$ and $\Psi(\infty) = 0$. □.

Results on boundary crossing probabilities for Brownian motion is studied by Lerche (1986) when the boundary is a continuous function. Sheike (1991) and Teunen and Goovaerts (1993) have obtained, using different motivations, some generalizations when the boundary is a discontinuous function.

3.2 A 'piecewise diffusion' Markov process

Let $(T_n, Z_n)_{n\geq 1}$ be a sequence of stochastic pairs representing a marked point process such that $T_1 < T_2 < \ldots$ are the non-negative points and $Z_1, Z_2 \ldots$ are the marks assumed to take values in some space $\mathcal{Z}$ endowed with a σ-algebra $\mathcal{E}$, see Brémaud (1981). The mark Z_i represents some phenomenon occurring at time T_i, for instance a claim amount. Consider now the process

$$R_t = R_0 + \int_0^t b(s, R_s)ds + \int_0^t \sigma(s, R_s)dB_s - \sum_{n=1}^{N_t} f(T_n, Z_n), \tag{3.16}$$

where b, σ are as above, $f : \mathcal{R}_+ \times \mathcal{Z} \to \mathcal{R}$ is some Borel measurable mapping and N_t is the number of jumps over $(0, t]$. We assume that there exists a unique solution to (3.16). In Møller (1993) we studied the case where $\sigma(t, r) \equiv 0$, which lead to PD (piecewise deterministic) processes. The jump process

$$X_t^{(f)} = \sum_{n=1}^{N_t} f(T_n, Z_n),$$

is also referred to as the risk process since it typically measures a total amount of random payments occurred over $(0, t]$. Concerning studies on the probability of ruin, special cases of (3.16) appear in Aase (1985) and Dufresne and Gerber (1991).

It is convenient to write $X_t^{(f)}$ as the stochastic double integral

$$X_t^{(f)} = \int_0^t \int_{\mathcal{Z}} f(s, z) dN_s(dz),$$

where $N_t(A)$ is the measure

$$N_t(A) = \sum_{i=1}^{\infty} I(T_i \leq t,\; Z_i \in A),$$

counting the number of jumps over $(0, t]$ with marks in $A \in \mathcal{E}$, and $I(F)$ denotes the indicator of a set in $F \in \mathcal{F}$. Also, we abbreviate $N_t = N_t(\mathcal{Z})$. The natural filtration is given by

$$\mathcal{F}_t^N = \sigma(N_s(A),\ s \le t,\ A \in \mathcal{E}),$$

and we assume that $N_t(A)$ forall $A \in \mathcal{E}$ admits a piecewise continuous $\mathcal{F}_t^N$-intensity defined informally by

$$\lambda_t(A)dt = E(dN_t(A) \,|\, \mathcal{F}_{t-}^N) + o(dt).$$

It can be more informative to write it as

$$\lambda_t(A) = \lambda_t \int_A G_t(dz), \quad \int_{\mathcal{Z}} G_t(dz) = 1,$$

where λ_t thus becomes the intensity of the point process N_t, and $G_t(dz)$ is interpreted as the conditional probability, given all prior information and that a jump occurred at time t, that the mark associated will belong to $(z, z+dz]$.

Assume that

$$C_t = \int_0^t \int_{\mathcal{Z}} f(s,z)\lambda_s(dz)ds$$

exists forall $t < \infty$. It is a continuous function and is called the compensator of $X_t^{(f)}$, implying that

$$M_t = X_t^{(f)} - C_t \tag{3.17}$$

becomes a zero mean martingale w.r.t. the natural filtration, see Brémaud (1981, p. 235).

The model in (3.16) opens the possibility to studying ruin probabilities in an economic environment where the force of interest is of Gaussian type: Namely, let the force of interest be governed by the Gaussian process

$$\delta_t = \mu(t) + \sigma(t)dB_t,$$

where $\mu(t)$ and $\sigma(t)$ are real-valued functions, and take then $\tilde{R}_t$ to be given by the stochastic differential equation

$$d\tilde{R}_t = (c(t) + \mu(t)\tilde{R}_t)dt + \sigma(t)\tilde{R}_t dB_t - \int_{\mathcal{Z}} f(t,z)dN_t(dz), \tag{3.18}$$

where $c(t)$ is playing the role of the premium rate in the model without interest, see also Example 3.4 below.

We can repeat the arguments in Subsection 3.1 and establish a partial integro-differential equation for evaluating $\Psi(t,r)$. To obtain the Markov property, we assume that $N_t(A)$ is a Poisson process for each A, implying that the intensity is deterministic and, in particular,

$$P(Z_n \in A \,|\, T_n = t) = \int_A G_t(dz), \quad \forall n.$$

Relation (3.3) will then modify to

$$\Psi(t \wedge \tau_D', Y_{t\wedge\tau_D'}) - \Psi(t', Y_{t'})$$

$$= \int_{t'}^{t\wedge\tau_D'} \frac{\partial\Psi}{\partial t}(s,R_s)ds + \int_{t'}^{t\wedge\tau_D'} \frac{\partial\Psi}{\partial r}(s,R_s)\left[b(s,R_s)ds + \sigma(s,R_s)dB_s\right]$$

$$+\frac{1}{2}\int_{t'}^{t\wedge\tau_D'} \sigma^2(s,R_s)\frac{\partial^2\Psi}{\partial r^2}(s,R_s)ds + \sum_{t'<s\le t\wedge\tau_D'} \{\Psi(s,R_s) - \Psi(s,R_{s-})\}, \tag{3.19}$$

where we for any $t > 0$ write

$$\sum_{0<s\le t}\{\Psi(s,R_s)-\Psi(s,R_{s-})\}=\int_0^t\int_{\mathcal{Z}}\{\Psi(s,R_{s-}-f(s,z))-\Psi(s,R_{s-})\}dN_s(dz). \tag{3.20}$$

Then the process

$$M_t^{**}=\int_{t'}^t I(\tau_D'\ge s)\{\Psi(s,R_{s-}-f(s,z))-\Psi(s,R_{s-})\}(dN_s(dz)-\lambda_s(dz)ds) \tag{3.21}$$

becomes a zero mean $\mathcal{F}_t^N\vee\mathcal{F}_t^B$-martingale over $[t',T]$ since the integrand is $\mathcal{F}_t^N\vee\mathcal{F}_t^B$-predictable, see Brémaud (1981, p. 235). Using (3.19)-(3.21), we obtain similar to (3.5) that

$$\begin{aligned}&\Psi(t\wedge\tau_D',Y_{t\wedge\tau_D'})-\Psi(t',Y_{t'})-M_t^*-M_t^{**}\\ &=\int_{t'}^{t\wedge\tau_D'}\frac{\partial\Psi}{\partial t}(s,R_s)ds+\int_{t'}^{t\wedge\tau_D'}\frac{\partial\Psi}{\partial r}(s,R_s)b(s,R_s)ds+\frac{1}{2}\int_{t'}^{t\wedge\tau_D'}\sigma^2(s,R_s)\frac{\partial^2\Psi}{\partial r^2}(s,R_s)ds\\ &\quad+\int_{t'}^{t\wedge\tau_D'}\int_{\mathcal{Z}}\{\Psi(s,R_{s-}-f(s,z))-\Psi(s,R_{s-})\}\lambda_s(dz)ds,\quad t\in[t',T],\end{aligned} \tag{3.22}$$

becomes a zero mean continuous $\mathcal{F}_t^N\vee\mathcal{F}_t^B$-martingale of bounded variation, and hence constant (zero).

We can then state:

Theorem 3.4 *Consider the process given by (3.16). For any fixed $t'\in[0,T)$, the process $\Psi(t\wedge\tau_D',R_{t\wedge\tau_D'})$ is a (bounded) martingale over $[t',T]$. Over the continuity points of $\sigma(t,r)$, $b(t,r)$, $\lambda_t(A)$ and f, the function $\Psi(t,r)$ satisfies the partial integro-differential equation*

$$\begin{aligned}&\frac{\partial\Psi}{\partial t}(t,r)+b(t,r)\frac{\partial\Psi}{\partial r}(t,r)+\frac{1}{2}\sigma^2(t,r)\frac{\partial^2\Psi}{\partial r^2}(t,r)\\ &=\lambda_t\Psi(t,r)-\int_{\mathcal{Z}}\Psi(t,r-f(t,z))\lambda_t(dz),\quad t\in(0,T),\ r\in D^c.\end{aligned} \tag{3.23}$$

Corollary 3.2 and 3.3 are modified analogously.

Ruin probabilities are obtained by taking $D=\{r|\,r\le 0\}$, and in a numerical implementation it would then be more convenient to introduce $\Phi=1-\Psi$ and make use of (2.1) ($\Phi(t,R_t)=0,\ R_t<0$) to obtain

$$\begin{aligned}&\frac{\partial\Phi}{\partial t}(t,r)+b(t,r)\frac{\partial\Phi}{\partial r}(t,r)+\frac{1}{2}\sigma^2(t,r)\frac{\partial^2\Phi}{\partial r^2}(t,r)\\ &=\lambda_t\Phi(t,r)-\int_{\{z|\,r\ge f(t,z)\}}\Phi(t,r-f(t,z))\lambda_t(dz),\quad t\in(0,T),\ r>0.\end{aligned} \tag{3.24}$$

Example 3.4. The probability of ruin in an economic environment with stochastic interest of Gaussian type and exponentially distributed claims. Consider the process R_t governed by the modified version of (3.18):

$$dR_t=(c+\mu R_t)dt+\sigma R_t dB_t-dX_t,$$

where c, μ and σ are constants and

$$X_t=\sum_{n=1}^{N_t}Z_n,$$

such that $Z_1, Z_2, \ldots$ represent the individual claim amounts.

The intensity of $N_t(A)$ is assumed to be given by

$$\lambda_t(dz) = \lambda\, G(dz),$$

implying that the $Z_1, Z_2 \ldots$ are assumed i.i.d. having distribution function G and independent of N_t, which becomes a homogeneous Poisson process with intensity $\lambda\,(> 0)$.

The R_t process is then a homogeneous Markov process, and by virtue of Theorem 3.4, the function $\Phi(r) = 1 - \Psi(r)$, where

$$\Psi(r) = P(\tau < \infty \,|\, R_0 = r),$$

satisfies the differential equation

$$(c + \mu r)\frac{d\Phi}{dr}(r) + \frac{1}{2}\sigma^2 r^2 \frac{d^2\Phi}{dr^2}(r)$$

$$= \lambda\Phi(r) - \lambda \int_{\{z\,|\,r \geq z\}} \Phi(r - z)G(dz), \quad r > 0. \tag{3.25}$$

Assuming that G is an exponential distribution with mean 1, say, we can, by differentiation on both sides in (3.25), obtain the third order differential equation

$$\frac{1}{2}\sigma^2 r^2 \frac{d^3\Phi}{dr^3}(r) + [c + \mu r + \sigma^2(\frac{1}{2}r^2 + r)]\frac{d^2\Phi}{dr^2}(r) + [\mu(r+1) + c - \lambda]\frac{d\Phi}{dr}(r) = 0.$$

□

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